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REPRESENTATION OF FRESNEL FUNCTIONS IN CONTINUED FRACTIONS

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Abstract. The goal of this paper is to provide an efficient method for computing the Fresnel functions by using the continued fractions with matrix arguments. We notice that the computation of these functions imposes many difficulties. Furthermore, we give some numerical examples which illustrated the theoretical results

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1. Introduction and motivation

Over the last two centuries, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. One of the main reasons why continued fractions are so useful in computation is that, in the convergence case, their expansions have the advantage that they converge more rapidly than other numerical algorithms [2, 6, 8]. So the extension of continued fractions theory from real numbers to the matrix and operator case has seen several development and interesting applications [3, 5, 11, 12].

The Fresnel integrals are fundamental in the theory of physics, they were originally used in the calculation of the electromagnetic field intensity in an environment where light bends around opaque objects. More recently, they have been used in the design of highways and railways, specifically their curvature transition zones, see track transition curve, [11].
In addition to generating the miraculous Cornu Spiral, the Fresnel integral can be used to solve the diffraction pattern of a light source through a close aperture. This is a consequence of the fact that circle waves are of the form of the light intensity,[1].

The Fresnel cosine and sine integral functions are defined by, (see [2])

$$C(z) = \int_0^z \cos(\frac{\pi}{2}t)^2 dt, \quad z \in \mathbb{C},$$

$$S(z) = \int_0^z \sin(\frac{\pi}{2}t)^2 dt, \quad z \in \mathbb{C}.$$

We have the symmetry properties

$$C(-z) = C(z), \quad S(-z) = -S(z),$$

$$\lim_{x\to+\infty} C(x) = \lim_{x\to+\infty} S(x) = 1/2.$$

Theses functions $C(z)$ and $S(z)$ are related to the error function by

$$C(z)+iS(z)=\frac{1+i}{2}\operatorname{erf}(\frac{\pi}{2}(1-i)z), C(z)-iS(z)=\frac{1+i}{2}\operatorname{erf}(\frac{\pi}{2}(1+i)z).$$

The series expansions of $C(x)$ and $S(x)$ are given by, (see [2])

$$C(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k(\pi/2)^{2k}}{(2k)!(4k+1)}x^{4k+1}, \quad x\in\mathbb{R}$$

and

$$S(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k(\pi/2)^{2k+1}}{(2k+1)!(4k+3)}x^{4k+3}, \quad x\in\mathbb{R}.$$

In this paper, we give the Fresnel integral functions representation as a sum of continued fractions in the real case and the matrix case.

2. Definitions and notations

Throughout this paper, $\mathcal{M}_m(\mathbb{R})$ will represent algebra of real matrices of sizes $m\times m$. Throughout this paper, we denote $\mathcal{M}_m$ instead of $\mathcal{M}_m(\mathbb{R})$.

For any matrices $A,B\in\mathcal{M}_m$ with B invertible, we write $A/B:=B^{-1}A$, in particular, if $A=I$, the identity matrix , then $I/B=B^{-1}$. It is easy to verify that for any invertible matrix X we have,

$$\frac{A}{B}=\frac{XA}{XB}.$$

Now, we introduce some topological notions of continued fractions with matrix arguments.
We provide $\mathcal{M}_m$ with the standard induced norm.

$$\forall A\in\mathcal{M}_m,\ \|A\|=Sup_{x\neq 0}\frac{\|Ax\|}{\|x\|}=Sup_{\|x\|=1}\|Ax\|.$$

Let (A_n) be a sequence of matrices in $\mathcal{M}_m$, we say that (A_n) converges in $\mathcal{M}_m$ if there exists a matrix $A\in\mathcal{M}_m$ such that $\|A_n-A\|$ tends to 0 when n tends to $+\infty$. In this case we write, $\lim_{n\to+\infty}A_n=A$.

Definition 2.1. *Let $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 1}$ be two sequences of matrices in $\mathcal{M}_m$. We denote the continued fraction expansion by*

$$A_0+\cfrac{B_1}{A_1+\cfrac{B_2}{A_2+\dots}}:=\left[A_0;\frac{B_1}{A_1},\frac{B_2}{A_2},\dots\right].$$

Sometimes, we denote this continued fraction by $\left[A_0;\frac{B_n}{A_n}\right]_{n=1}^{+\infty}$ *or* $K\left(B_n/A_n\right)$, *where*

$$\left[A_0;\frac{B_i}{A_i}\right]_{i=1}^{n}=\left[A_0;\frac{B_1}{A_1},\frac{B_2}{A_2},\dots,\frac{B_n}{A_n}\right].$$

The fractions $\frac{P_n}{Q_n} = \left[A_0; \frac{B_i}{A_i}\right]_{i=1}^{n}$ *is called the the* n^{th} *convergent of the continued fraction* $K(B_n/A_n)$.
The continued fraction $\left[A_0, \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ *is said to be convergent in* $\mathcal{M}_m$ *if the sequence* $(P_n/Q_n)_n = (Q_n^{-1}P_n)_n$ *converges in* $\mathcal{M}_m$ *in the sense that there exists a matrix* $F \in \mathcal{M}_m$ *such that* $\lim_{n\to+\infty} \|F_n - F\| = 0$. *In this case, we denote*

$$F = \left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}.$$

We note that the evaluation of the n^{th} convergent according to Definition **??** is not practical because we have to repeatedly invert matrices. The following proposition gives an adequate method to calculate $K(B_n/A_n)$.

Proposition 2.2. *For the continued fraction* $K(B_n/A_n)$, *define*

$$\begin{cases} P_{-1} = I, \ P_0 = A_0 \\ Q_{-1} = 0, \ Q_0 = I \end{cases} \quad \text{and} \quad \begin{cases} P_n = A_n\, P_{n-1} + B_n P_{n-2} \\ Q_n = A_n\, Q_{n-1} + B_n Q_{n-2} \end{cases} \quad n \geq 1.$$

Then the matrix P_n/Q_n *is the* n^{th} *convergent of* $K(B_n/A_n)$.

Proof. This can be done by induction. □

Theorem 2.3. *[11]. Let* (A_n), (B_n) *be two sequences of* $\mathcal{M}_m$. *If*

$$\| (B_{2k-2}...B_2)^{-1} A_{2k-1}^{-1} B_{2k-1}...B_1 \| \leq \alpha$$

and

$$||(B_{2k-1}...B_1)^{-1} A_{2k}^{-1} B_{2k}...B_2|| \leq \beta$$

for all $k \geq 1$, *where* $0 < \alpha < 1, 0 < \beta < 1$ *and* $\alpha\beta \leq 1/4$, *then the continued fraction* $K(B_n/A_n)$ *converges in* $\mathcal{M}_m$.

To end this section, We need to present the following Propositions.

Proposition 2.4. *[8]. Let* $C \in \mathcal{M}_m$ *such that* $||C|| < 1$, *then the matrix* $I - C$ *is invertible and we have*

$$||(I - C)^{-1}|| \leq \frac{1}{1 - ||C||}.$$

Proposition 2.5. *[2]. If the function* $f(x)$ *can be expanded in a power series in the interval* $| x - x_0 | < r$, *as*

$$f(x) = \sum_{p=0}^{+\infty} \alpha_p (x - x_0)^p,$$

then this expansion remains valid when the scalar argument x *is replaced by a matrix* A *whose characteristic value belong to the interval of convergence.*

3. Main results

3.1. Continued fractions expansions of Fresnel integral functions. This section is devoted to give representations of the Fresnel integral functions as a sum of continued fractions in real and matrix cases.

3.1.1. *Real case.*

Theorem 3.1. *(i) Let x be a real number, a representation of Fresnel cosine integral function as a sum of continued fractions is*

$$C(x) = \left[x; \frac{-(\pi/2)^2 x^5}{10}, \frac{(\pi/2)^2 A_{2n-2}^2 (4n-3)^2 x^4}{(4n+1)A_{2n}^2 - (\pi/2)^2 (4n-3) x^4}\right]_{k=2}^{+\infty}.$$

(ii) Let x be a real number, a representation of Fresnel sine integral function as a sum of continued fractions is

$$S(x) = \left[\frac{\pi}{6}x^3; \frac{-(\pi/2)^3 x^7}{42}, \frac{(\pi/2)^2 A_{2n-1}^2 (4n-1)^2 x^4}{(4n+3)A_{2n+1}^2 - (\pi/2)^2 (4n-1) x^4}\right]_{k=2}^{+\infty}.$$

In order to prove Theorem 3.1, we give the following lemmas.

Lemma 3.2. *[6] Let g be a function with Taylor series expansion in an interval $I \subset \mathbb{R}$ is*

$$g(y) = \sum_{k=0}^{+\infty} b_k y^k.$$

Then, the development in continued fractions of $g(y)$ is given by

$$g(y) = \left[b_0; \frac{b_1 y}{1}, \frac{-b_2 y}{b_1 + b_2 y}, \frac{-b_1 b_3}{b_2 + b_3 y}, \frac{-b_{n-2} b_n y}{b_{n-1} + b_n y}\right]_{n=4}^{+\infty}.$$

The following lemma characterizes equivalence of continued fractions.

Lemma 3.3. *[5]. Let (r_n) be a non-zero sequence of real numbers. Then, one has*

$$\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, ..., \frac{b_n}{a_n}, ...\right] = \left[a_0; \frac{r_1 b_1}{r_1 a_1}, \frac{r_2 r_1 b_2}{r_2 a_2}, ..., \frac{r_n r_{n-1} b_n}{r_n a_n}, ...\right].$$

We recall that the cosine Fresnel integral function is given by

$$C(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{4k+1} (\frac{\pi}{2})^{2k} \frac{x^{4k+1}}{(2k)!}$$

and

$$S(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{4n+3} (\frac{\pi}{2})^{2n+1} \frac{x^{4n+3}}{(2n+1)!}$$

Proof. (of theorem 3.1.)

(i) The Fresnel integral function $C(x)$ can be written as

$$C(x)=x\sum_{n=0}^{+\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!(4n+1)}x^{4n},\quad b_n=\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!(4n+1)}.$$

Let $y=x^4$, from the last formulae, we deduce

$$b_0=1,\ \frac{b_1y}{1}=\frac{\frac{-(\pi/2)^2y}{5.2!}}{1},\ \frac{-b_2y}{b_1+b_2y}=\frac{\frac{-(\pi/2)^4y}{9.4!}}{b_1+\frac{(\frac{\pi}{2})^4y}{4!9}}.$$

Let $A_n^k=\dfrac{n!}{(n-k)!}$, for all $k\leq n$. For $n\geq 3$, we get

$$b_{n-2}b_ny=\frac{(-1)^{n-2}(\frac{\pi}{2})^{2n-4}}{(2n-4)!(4n-7)}.\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!(4n+1)}y=\frac{(\frac{\pi}{2})^{4n-4}}{((2n-4)!)^2A_{2n}^4(4n-7)(4n+1)}y.$$

Furthermore, we have

$$\begin{aligned}b_{n-1}+b_ny&=\frac{(-1)^{n-1}(\pi/2)^{2n-2}}{(2n-2)!(4n-3)}+\frac{(-1)^n(\pi/2)^{2n}}{(2n)!(4n+1)}y\\&=\frac{(-1)^{n-1}(\pi/2)^{2n-2}}{(2n)!(4n+1)(4n-3)}\left(A_{2n}^2(4n+1)-(\pi/2)^2(4n-3)y\right).\end{aligned}$$

Then, we obtain

$$\frac{-b_{n-2}b_ny}{b_{n-1}+b_ny}=\frac{\frac{-(\pi/2)^{4n-4}y}{((2n-4)!)^2A_{2n}^4(4n-7)(4n+1)}}{\frac{(-1)^{n-1}(\pi/2)^{2n-2}}{(2n)!(4n+1)(4n-3)}\left(A_{2n}^2(4n+1)-(\pi/2)^2(4n-3)y\right)}.$$

Therefore, the continued fraction expansion of $C(x)$ is

$$C(x)=x\left[1;\frac{\frac{-(\pi/2)^2}{2!5}y}{1},\frac{\frac{-(\pi/2)^4x^4}{4!9}}{\frac{-(\pi/2)^2}{2!5}+\frac{(\pi/2)^4y}{4!9}},\right.$$
$$\left.\frac{\frac{-(\pi/2)^{4n-4}y}{((2n-4)!)^2A_{2n}^4(4n-7)(4n+1)}}{\frac{(-1)^{n-1}(\pi/2)^{2n-2}}{(2n)!(4n+1)(4n-3)}\left(A_{2n}^2(4n+1)-(\pi/2)^2(4n-3)y\right)}\right]_{n=3}^{+\infty}$$

Let us define the sequence $(r_n)_{n\geq 1}$ by

$$\begin{cases}r_1=10,\\ r_n=\dfrac{(4n-3)(2n)!(4n+1)}{(-1)^{n-1}(\pi/2)^{2n-2}},\quad n\geq 2.\end{cases}$$

Then, we found

$$\begin{cases} \dfrac{r_1 c_1}{r_1 d_1} = \dfrac{-(\pi/2)^2 y}{10}, \\ \dfrac{r_n r_{n-1} c_n}{r_n d_n} = \dfrac{(\pi/2)^2 A_{2n-2}^2 (4n-3)^2 y}{(4n+1)A_{2n}^2 - (\pi/2)^2 (4n-3) y}, \ n \geq 2. \end{cases}$$

Finally, we obtain

$$C(x) = \left[x; \frac{-(\pi/2)^2 x^5}{10}, \frac{(\pi/2)^2 A_{2n-2}^2 (4n-3)^2 y}{(4n+1)A_{2n}^2 - (\pi/2)^2 (4n-3) y} \right]_{k=2}^{+\infty},$$

which finish the proof of (i).

(ii) We prove it by a similar method of (i). We have

$$S(x) = x^3 \sum_{n=0}^{+\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!(4n+3)} x^{4n}, \ \ b'_n = \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!(4n+3)}.$$

So, we have

$$b'_0 = \frac{\pi}{6}, \ \ \frac{b'_1 y}{1} = \frac{\frac{-(\pi/2)^3}{7.3!} y}{1}, \ \ \frac{-b'_2 y}{b'_1 + b'_2 y} = \frac{\frac{-(\pi/2)^5 y}{11.5!}}{\frac{-(\pi/2)^3}{7.3!} + \frac{(\pi/2)^5 y}{11.5!}} = \frac{\frac{-(\pi/2)^5 y}{11.5!}}{\frac{-(\pi/2)^3}{77.5!}(11A_5^2 - 7(\pi/2)^2 y)}.$$

So, for all $n \geq 3$, we get

$$\frac{-b'_{n-2} b'_n y}{b'_{n-1} + b'_n y} = \frac{\frac{-(\pi/2)^{4n-2} y}{(2n+1)!(2n-3)!(4n-5)(4n+3)}}{\frac{(-1)^{n-1}(\pi/2)^{2n-1}}{(2n+1)!(4n-1)(4n+3)} \left(A_{2n+1}^2 (4n+3) - (4n-1)(\pi/2)^2 y \right)}.$$

Therefore, the continued fraction expansion of $S(x)$ becomes

$$S(x) = x^3 \left[\frac{\pi}{6}; \frac{\frac{-(\pi/2)^3}{7.3!} x^4}{1}, \frac{\frac{-(\pi/2)^5 y}{11.5!}}{\frac{-(\pi/2)^3}{77.5!}(11A_5^2 - 7(\pi/2)^2 y)}, \right.$$
$$\left. \frac{\frac{-(\pi/2)^{4n-2} y}{(2n+1)!(2n-3)!(4n-5)(4n+3)}}{\frac{(-1)^{n-1}(\pi/2)^{2n-1}}{(2n+1)!(4n-1)(4n+3)} \left(A_{2n+1}^2 (4n+3) - (4n-1)(\pi/2)^2 y \right)} \right]_{n=3}^{+\infty}$$

In order to simplify the previous expansion of $S(x)$, let

$$r'_1 = 7.3!, \ \ r'_n = \frac{(2n+1)!(4n+3)(4n-1)}{(-1)^{n-1}(\pi/2)^{2n-1}}, \ n \geq 2.$$

By applying the Lemma 3.3, with the following transformations

$$\begin{cases} \dfrac{r'_1 C'_1}{r'_1 d'_1} = \dfrac{-(\pi/2)^3 x^4}{7.3!}, \\ \dfrac{r'_n r'_{n-1} c'_n}{r'_n d'_n} = \dfrac{A^2_{2n-1}(4n-1)^2(\pi/2)^2 x^4}{A^2_{2n+1}(4n+3)-(\pi/2)^2(4n-1)x^4} \end{cases}$$

we complete the proof of (ii).

□

3.1.2. *Matrix case.*

Definition 3.4. *Let A be a matrix in $\mathcal{M}_m$, we define the Fresnel integral functions of matrix A by the expressions*

$$C(A) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{4k+1} (\frac{\pi}{2})^{2k} \frac{A^{4k+1}}{(2k)!}$$

and

$$S(A) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{4n+3} (\frac{\pi}{2})^{2n+1} \frac{A^{4n+3}}{(2n+1)!}.$$

Theorem 3.5. *Let A be a matrix of M_m such that $\| A \| = \alpha$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < \frac{1}{2}$.*

(i) The matrix continued fraction

$$\left[A; \frac{-(\pi/2)^2 A^5}{10I}, \frac{(\pi/2)^2 A^2_{2n-2}(4n-3)^2 A^4}{(4n+1)A^2_{2n} I - (\pi/2)^2(4n-3)A^4}\right]_{n=2}^{+\infty},$$

converge in $\mathcal{M}_m$. Furthermore, this continued fraction represents the function $C(A)$. So, we have

$$C(A) = \left[A; \frac{-(\pi/2)^2 A^5}{10I}, \frac{(\pi/2)^2 A^2_{2n-2}(4n-3)^2 A^4}{(4n+1)A^2_{2n} I - (\pi/2)^2(4n-3)A^4}\right]_{n=2}^{+\infty}$$

(ii) The matrix continued fraction

$$\left[\frac{\pi}{6}A^3; \frac{-(\pi/2)^3 A^7}{42I}, \frac{(\pi/2)^2 A^2_{2n-1}(4n-1)^2 A^4}{(4n+3)A^2_{2n+1} I - (\pi/2)^2(4n-1)A^4}\right]_{n=2}^{+\infty}.$$

converge in $\mathcal{M}_m$. Furthermore, this continued fraction represents the function $C(A)$. So, we have

$$S(A) = \left[\frac{\pi}{6}A^3; \frac{-(\pi/2)^3 A^7}{42I}, \frac{(\pi/2)^2 A^2_{2n-1}(4n-1)^2 A^4}{(4n+3)A^2_{2n+1} I - (\pi/2)^2(4n-1)A^4}\right]_{n=2}^{+\infty}.$$

Proof. (i) We keep the same notations as in Theorem 3.5. In order to prove the convergence of the continued fraction $K(C_k/D_k)$ with

$$\begin{cases} C_1 &= -(\pi/2)^2 A^5, \\ D_1 &= 10I, \end{cases}$$

and for $k \geq 2$,

$$\begin{cases} C_k &= (\pi/2)^2 A_{2k-2}^2 (4k-3)^2 A^4, \\ D_k &= (4k+1) A_{2k}^2 I - (\pi/2)^2 (4k-3) A^4. \end{cases}$$

We should verify that the conditions of Theorem 2.3 are satisfied. One has

$$\|C_1 C_2^{-1}\| = \| \frac{-(\pi/2)^2}{(\pi/2)^2 A_2^2 5^2} A^5 A^{-4} \| \leq \|A\|,$$

for $k \geq 1$, we obtain

$$\|C_{2k-1} C_{2k}^{-1}\| = \| \frac{(\pi/2)^2 A_{4k-3}^2 (8k-7)^2}{(\pi/2)^2 A_{4k-2}^2 (8k-3)^2} A^4 A^{-4} \| \leq 1$$

and

$$\begin{aligned} \|D_{2k-1}^{-1}\| &= \|((8k-3) A_{4k-2}^2 I - (\pi/2)^2 (8k-7) A^4)^{-1}\| \\ &= \frac{1}{(8k-3) A_{4k-2}^2} \|(I - \frac{(\pi)^2 (8k-7)}{(8k-3) A_{4k-2}^2} A^4)^{-1}\| \\ &\leq \|(I - \frac{(\pi/2)^2 (8k-7)}{(8k-3) A_{4k-2}^2} A^4)^{-1}\|. \end{aligned}$$

Since the factors of the product $(C_{2k-2} ... C_2)^{-1} D_{2k-1}^{-1} (C_{2k-3} ... C_1)$ commute between them, so

$$\begin{aligned} \|(C_{2k-2} ... C_2)^{-1} D_{2k-1}^{-1} (C_{2k-3} ... C_1)\| &= \|(C_1 C_2^{-1}) ... (C_{2k-3} C_{2k-2}^{-1}) D_{2k-1}^{-1}\| \\ &\leq \|A(I - \frac{(\pi/2)^2 (8k-7)}{(8k-3) A_{4k-2}^2} A^4)^{-1}.\| \end{aligned}$$

According to Proposition 2.4 and the fact that

$$\lim_{k \to +\infty} \frac{(\pi/2)^2 (8k-7)}{(8k-3) A_{4k-2}^2} \mid\mid A^4 \mid\mid = 0,$$

we obtain for all sufficiently large k

$$\begin{aligned} \|(C_{2k-2} ... C_2)^{-1} D_{2k-1}^{-1} (C_{2k-3} ... C_1)\| &\leq \|A\| \|(I - \frac{(\pi/2)^2 (8k-7)}{(8k-3) A_{4k-2}^2} A^4)^{-1}\| \\ &\leq \|A\| \frac{1}{1 - \frac{(\pi/2)^2 (8k-7)}{(8k-3) A_{4k-2}^2} \|A^4\|} \\ &\leq \|A\| \leq \frac{1}{2}. \end{aligned}$$

To prove the second inequality of Theorem 2.3, we have

$$\begin{aligned}
\|(C_{2k-1}...C_3)^{-1}D_{2k}^{-1}(C_{2k}...C_2)\| &= \|(C_2C_3^{-1})...(C_{2k-2}C_{2k-1}^{-1})C_{2k}D_{2k}^{-1}\| \\
&\leq \frac{(\pi/2)^2A_{4k-2}^2(8k-3)^2}{(8k+1)A_{4k}^2}\|A^4(I-\frac{(\pi/2)^2(8k-3)}{(8k+1)A_{4k}^2}A^4)^{-1}\| \\
&\leq \|A^4\|\frac{1}{1-\dfrac{(\pi)^2(8k-3)}{(8k+1)A_{4k}^2}\|A^4\|} \\
&\leq \|A\|^4\leq \frac{1}{2}
\end{aligned}$$

which complete the proof of (i). We study the convergence of the continued fraction $K(C_k'/D_k')$ with

$$\begin{cases} C_1' &= -(\pi/2)^3A^7, \\ D_1' &= 42I, \end{cases}$$

and for all $k \geq 2$,

$$\begin{cases} C_k' &= (\pi/2)^2A_{2k-1}^2(4k-1)^2A^4, \\ D_k' &= (4k+3)A_{2k+1}^2I-(\pi/2)^2(4k-1)A^4. \end{cases}$$

We prove (ii) by the same method as below. We have

$$\begin{aligned}
\|(C_{2k-2}'...C_2')^{-1}D_{2k-1}'^{-1}(C_{2k-3}'...C_1')\| &= \|(C_1'C_2'^{-1})...(C_{2k-3}'C_{2k-2}'^{-1})D_{2k-1}'^{-1}\| \\
&\leq \frac{1}{(8k-1)A_{4k-1}^2}\|A^3(I-\frac{(\pi/2)^2(8k-5)}{(8k-1)A_{4k-1}^2}A^4)^{-1}\| \\
&\leq \|A^3\|\frac{1}{1-\dfrac{(\pi/2)^2(4k-5)}{(8k-1)A_{4k-1}^2}\|A^4\|} \\
&\leq \|A\|^3\leq \frac{1}{2}.
\end{aligned}$$

We also have

$$\begin{aligned}
\|(C_{2k-1}'...C_3')^{-1}D_{2k}'^{-1}(C_{2k}'...C_2')\| &= \|(C_2'C_3'^{-1})...(C_{2k-2}'C_{2k-1}'^{-1})C_{2k}'D_{2k}'^{-1}\| \\
&\leq \frac{(\pi/2)^2A_{4k-1}^2(8k-1)^2}{(8k+3)A_{4k+1}^2}\|A^4(I-\frac{(\pi/2)^2(8k-1)}{(8k+3)A_{4k+1}^2}A^4)^{-1}\| \\
&\leq \|A^4\|\frac{1}{1-\dfrac{(\pi/2)^2(8k-1)}{(8k+3)A_{4k+1}^2}\|A^4\|} \\
&\leq \|A\|^4\leq \frac{1}{2}.
\end{aligned}$$

hence the proof of (ii).

□

4. Numerical applications

In this section, we present some numerical examples of our theoretical results, beginning by real case

Example 4.1.

Table 1. The following table clarifies the differences between the Fresnel integral function $C(x)$ and its convergents when Applying the (i) of theorem 3.1.

x	$(C-F_1)(x)$	$(C-F_2)(x)$	$(C-F_3)(x)$	$(C-F_4)(x)$	$(C-F_5)(x)$
0.05	-5.50e-14	1.95e-20	-4.12e-27	5.72e-34	-5.61e-41
0.1	-2.81e-11	1.60e-16	-5.40e-22	1.20e-27	-1.88e-33
0.2	-1.44e-8	1.31e-12	-7.08e-17	2.51e-21	-6.32e-26
0.5	-5.48e-5	1.95e-7	-4.11e-10	5.71e-13	-5.61e-16
0.75	-2.07e-3	3.77e-5	-4.03e-7	2.84e-9	-1.41e-11
1	-2.66e-2	1.55e-3	-5.28e-5	1.18e-6	-1.86e-8

Example 4.2.

Table 2. The following table clarifies the differences between the Fresnel integral function $S(x)$ and its convergents when applying the (ii) of theorem .1.

x	$(S-F_1)(x)$	$(S-F_2)(x)$	$(S-F_3)(x)$	$(S-F_4)(x)$	$(S-F_5)(x)$
0.05	-3.53e-17	9.52e-24	-1.61e-30	1.86e-37	-1.57e-44
0.1	-7.24e-14	3.12e-19	-8.44e-25	1.56e-30	-2.10e-36
0.2	-1.48e-10	1.02e-14	-4.42e-19	1.31e-23	-2.82e-28
0.5	-3.52e-6	9.50e-9	-1.60e-11	1.86e-14	-1.56e-17
0.75	-3.01e-4	4.13e-6	-3.54e-8	2.08e-10	-8.89e-13
1	-6.94e-3	3.03e-4	-8.28e-6	1.54e-7	-2.08e-9

Now we pass to the matrix case, we illustrate the theoretical results obtained in the theorem 3.5, we start with an example of the matrix function $C(A)$

Example 4.3. *let A be an 3×3 matrix such that*

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{4} \end{pmatrix}$$

The norm of A is $\|A\| = 0.4375$, it is of course strictly less than $1/2$. By applying the theoretical results obtained in the theorems 3.4 the difference *between the fresnel integral function of matrix $C(A)$ and $S(A)$ and its convergents respectably is given by*

$$(F_1 - C)(A) = \begin{pmatrix} -4.58e-6 & -3.35e-6 & -4.58e-6 \\ -3.35e-6 & -2.47e-6 & -3.35e-6 \\ -4.58e-6 & -3.35e-6 & -4.58e-6 \end{pmatrix},$$

$$(F_2 - C)(A) = \begin{pmatrix} 8.18e-9 & 5.99e-9 & 8.18e-9 \\ 5.99e-9 & 4.38e-9 & 5.99e-9 \\ 8.18e-9 & 5.99e-9 & 8.18e-9 \end{pmatrix},$$

$$(F_3 - C)(A) = \begin{pmatrix} -8.65e-12 & -6.33e-12 & -8.65e-12 \\ -6.33e-12 & -4.63e-12 & -6.33e-12 \\ -8.65e-12 & -6.33e-12 & -8.65e-12 \end{pmatrix},$$

$$(F_4 - C)(A) = \begin{pmatrix} 6.01e-15 & 4.40e-15 & 6.01e-15 \\ 4.40e-15 & 3.22e-15 & 4.40e-15 \\ 6.01e-15 & 4.40e-15 & 6.01e-15 \end{pmatrix},$$

$$(F_5 - C)(A) = \begin{pmatrix} -2.96e-18 & -2.16e-18 & -2.96e-18 \\ -2.16e-18 & -1.58e-18 & -2.16e-18 \\ -2.96e-18 & -2.16e-18 & -2.96e-18 \end{pmatrix}.$$

$$(F_1 - S)(A) = \begin{pmatrix} -2.08e-7 & -1.52e-7 & -2.08e-7 \\ -1.52e-7 & -1.12e-7 & -1.52e-7 \\ -2.08e-7 & -1.52e-7 & -2.08e-7 \end{pmatrix},$$

$$(F_2 - S)(A) = \begin{pmatrix} 2.81e-10 & 2.06e-10 & 2.81e-10 \\ 2.06e-10 & 1.51e-10 & 2.06e-10 \\ 2.81e-10 & 2.06e-10 & 2.81e-10 \end{pmatrix},$$

$$(F_3 - S)(A) = \begin{pmatrix} -2.39e-13 & -1.75e-13 & -2.39e-13 \\ -1.75e-13 & -1.28e-13 & -1.75e-13 \\ -2.39e-13 & -1.75e-13 & -2.39e-13 \end{pmatrix},$$

$$(F_4 - S)(A) = \begin{pmatrix} 1.38e-16 & 1.01e-16 & 1.38e-16 \\ 1.01e-16 & 7.44e-17 & 1.01e-16 \\ 1.38e-16 & 1.01e-16 & 1.38e-16 \end{pmatrix},$$

$$(F_5 - S)(A) = \begin{pmatrix} -5.86e-20 & -4.29e-20 & -5.86e-20 \\ -4.29e-20 & -3.14e-20 & -4.29e-20 \\ -5.86e-20 & -4.29e-20 & -5.86e-20 \end{pmatrix}.$$

The results established above show that the continued fraction algorithm converges very quickly. Indeed for the integral cosine C(A) we gained 5 places since the first iteration and so on. This shows the importance of this approach.

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